

The order encoding: from tractable CSP to tractable SAT

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Many mathematical and practical problems can be expressed as *constraint satisfaction problems* (CSPs). The general CSP is known to be NP-complete, but many different conditions have been identified which are sufficient to ensure that classes of instances satisfying those conditions are tractable, that is, solvable in polynomial time [1–4, 7]. The increasing efficiency of SAT-solvers has led to the development of SAT-based constraint solvers and various SAT encodings for CSPs [6]. However, most previous comparison between such encodings has been purely empirical. In a recent paper we showed that current SAT-solvers will decide the satisfiability of the *direct encoding* of any CSP instance with bounded width in expected polynomial time [5]. In this paper we give a theory-based argument to prefer the *order encoding* instead for certain other families of tractable constraint satisfaction problems. We consider problems of the form $\text{CSP}(C)$, consisting of all CSP instances whose constraint relations belong to some fixed set of relations C , known as a *constraint language*. Schaefer’s well-known dichotomy theorem [7] identifies all the tractable constraint languages over a Boolean domain, that is, all the tractable language classes for SAT.

A *sparse encoding* of a CSP instance introduces a new Boolean variable, $x_{v,a}^-$, for each possible variable assignment, $v = a$. The *log encoding* introduces a Boolean variable for each *bit* in the value of a CSP variable. It turns out that under such encodings tractable CSPs cannot be translated into tractable language classes of SAT. In particular, we have shown that:

Proposition 1 *No sparse encoding of a CSP instance with domain size > 2 belongs to a tractable language class of SAT. Moreover, the log encoding of any CSP instance with domain size > 7 containing certain unary constraints does not belong to any tractable language class of SAT.*

In the order encoding [8] each Boolean variable, $x_{v,c}^{\leq}$, represents a comparison, $v \leq c$. Under that encoding we have shown that certain tractable CSP classes are translated to tractable language classes of SAT, and hence efficiently solvable.

For example, a CSP instance is called *constant-closed* if every constraint in it allows some fixed constant value d to be assigned to all variables in its scope.

Theorem 1 *If all the constraints in a CSP instance are constant-closed for the lowest domain value, then its order encoding will be constant-closed for the value True.*

Hence, we have shown that using the order encoding to translate a CSP instance that is constant-closed for the lowest domain value gives a set of clauses satisfying

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the first condition of Schaefer's Dichotomy Theorem. Similarly, constraints that are constant-closed under the highest domain value translate under the order encoding to clauses that satisfy the second condition of that theorem.

A rather more interesting family of tractable constraint satisfaction problems is the class of CSPs whose constraints are all *max-closed*.

Lemma 1 ([4]) *If the domain of the variables is $\{\text{True}, \text{False}\}$, with $\text{False} < \text{True}$, then a constraint is min-closed if and only if it is logically equivalent to a conjunction of Horn clauses over literals representing comparisons.*

Theorem 2 *If a CSP instance P contains max-closed constraints only, then its order encoding will be min-closed.*

Hence, max-closed constraints translate using the order encoding to clauses satisfying the third condition of Schaefer's Dichotomy Theorem. By symmetry between min-closed and max-closed constraints, min-closed constraints translate to clauses satisfying the fourth condition of Schaefer's Dichotomy Theorem.

Connected-row-convex constraints were first defined in [3] using a standard matrix representation of binary relations. Here is an alternative characterisation:

Lemma 2 ([2]) *A constraint is connected-row-convex if and only if it is logically equivalent to a conjunction of 2-CNF clauses over literals representing comparisons.*

Connected-row-convex constraints translate to clauses satisfying the fifth condition of Schaefer's Dichotomy Theorem due to the following result:

Theorem 3 *If a CSP instance P contains only connected-row-convex constraints, then its order encoding will be connected-row-convex.*

The final, sixth, condition in Schaefer's Dichotomy Theorem can never be satisfied using the order encoding, since (for all domains with 3 or more elements) it is already broken by the consistency clauses, $\neg(x_{v,c-1}^{\leq}) \vee (x_{v,c}^{\leq})$. Hence we have given a complete list of all constraint languages which are encoded to tractable language classes for SAT using the order encoding.

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